Topics in complex analysis (Lecture 6) Oct 28, 2024

Lemma 4.1. *It holds that $|E_n(z) - 1| \le |z|^{n+1}$ for all $z \in B_1(0)$.*

<u>Proof</u> Recall $E_0(z) = 1-z$, $E_n(z) = (1-z)\, e^{\sum_{k=1}^{n} z^k/k}$

Since $|E_0(z) - 1| = |z|$ the claim holds for $n = 0$.

For general $n \ge 1$ we write for simplicity $p_n(z) := \sum_{k=1}^{n} \frac{z^k}{k}$.

Differentiate: $(1-z)\, p_n'(z) = (1-z) \sum_{j=0}^{n-1} z^j = 1 - z^n$ (telescopic sum

hence

$$E_n'(z) = \frac{d}{dz} (1-z)\, e^{p_n(z)}$$

Leibniz $= -e^{p_n(z)} + \underbrace{(1-z)\, p_n'(z)}_{= 1 - z^n} e^{p_n(z)} = -z^n e^{p_n(z)}$

On the other hand denoting by $E_n(z) = \sum_{k \in \mathbb{N}_0} a_k z^k$ the Taylor series of E_n around 0:

$$-z^n e^{p_n(z)} = E_n'(z) = \sum_{k \in \mathbb{N}_0} k a_k z^{k-1}$$

[For $|z|$ small $|z| \le |z|^{42}$ not true]

The rhs has a zero of order n in 0! Hence $a_k = 0 \quad \forall\, 1 \le k \le n.$ (*)

Now $E_n(0) = 1 = a_0$ (by Taylor series rep of E_n) so

$$|E_n(z) - 1| = \Big| 1 + \sum_{k \ge n+1} a_k z^k - 1 \Big|$$

(*)

$$\le \sum_{k \ge n+1} |a_k|\, |z|^k \le \underbrace{\sup_{k \ge n+1} |z|^k}_{= |z|^{n+1} \text{ because } |z| \le 1} \overbrace{\sum_{k \ge n+1} |a_k|}^{\overset{(\#)}{=} 1} = |z|^{n+1}$$

□

(#) $0 = E_n(1) = 1 + \sum_{k \ge n+1} a_k$

$a_k = -|a_k|$ because the Taylor coeff of e^{p_n} are nonneg.

Theorem 4.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers that is discrete in $\mathbb{C}$. For each $n \in \mathbb{N}$ set $o_n := \#\{k \in \mathbb{N} : a_k = a_n\}$. Assume that $a_n \neq 0$ and $o_n < +\infty$ for all $n \in \mathbb{N}$. Then*

$$f(z) := \prod_{n=1}^{\infty} E_n\left(\tfrac{z}{a_n}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n}$$

converges locally normally (1) and defines an entire function (2) with $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$. (3)

Proof (1) We first show local normal cvg (which is equiv to: $\forall K \subseteq \mathbb{C}$ cpt $\displaystyle\sum_{k=1}^{\infty} \sup_{z\in K} \left| E_n\left(\frac{z}{a_n}\right) - 1 \right| < \infty$)

originally a nbhd of a pt

Let $K \subseteq \mathbb{C}$ cpt. Discreteness $\Longrightarrow$ $|a_n| \longrightarrow \infty$ (because otherwise $\{a_n : n \in \mathbb{N}\}$ would have an acc pt)

Since $|z|$ is bdd in $z \in K$, $\displaystyle\sup_{z\in K} \left|\frac{z}{a_n}\right| \leq \frac{c}{|a_n|} \longrightarrow 0$

In particular $\exists n_0 \in \mathbb{N}$: $\displaystyle\sup_{z\in K}\left|\frac{z}{a_n}\right| \leq \frac{1}{2}$. In part. $\forall n \geq n_0$: $z/a_n \in B_1(0)$ hence Lemma 4.1 applies:

$$\sum_{n=n_0}^{\infty} \sup_{z\in K} \left| E_n\left(\frac{z}{a_n}\right) - 1 \right| \leq \sum_{n=n_0} \underbrace{\sup_{z\in K} \left|\frac{z}{a_n}\right|^{n+1}}_{\leq 2^{-n-1}} < \infty$$

summability of $\sum_k 2^{-k}$

Hence $\prod_n E_n$ cvg loc. normally.

(2) Since E_n is entire, so is their loc. normally cvg product by Cor. 3.7.

(3) By Lemma 3.11 we obtain

$$Z(f) = \bigcup_{n=1}^{\infty} Z\left(E_n\left(\frac{\cdot}{a_n}\right)\right) = \{a_n : n \in \mathbb{N}\}$$

$$o_{a_n}(f) = \sum_{j=1}^{\infty} \underbrace{o_{a_n}\left(E_j\left(\frac{\cdot}{a_j}\right)\right)}_{=1 \text{ if } a_n = a_j,\ 0 \text{ otherwise}} = o_n$$

□

Corollary 4.4. *Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function such that $g \not\equiv 0$ and write its zeros in $\mathbb{C} \setminus \{0\}$ as*

$$(\underbrace{a_1, \dots, a_1}_{o_{a_1}(g) \text{ times}}, \underbrace{a_2, \dots, a_2}_{o_{a_2}(g) \text{ times}}, \dots) =: (s_1, s_2, s_2, \dots) = s$$

Then we can write

$$g(z) = e^{h(z)} z^{o_0(g)} \prod_{n=1}^{\dim(s)} E_n\left(\tfrac{z}{s_n}\right),$$

where $h : \mathbb{C} \to \mathbb{C}$ is an entire function.

<u>Proof</u> Apply Weierstraß prod thm to the given zeros. This yields

$$f(z) = z^{o_0(g)} \prod_{n=1}^{\dim s} E_n\left(\frac{z}{s_n}\right) \quad \text{is an entire fct with}$$

$$Z(f) = Z(g) \quad \& \quad o_z(f) = o_z(g) \quad \forall z \in \mathbb{C}$$

Claim: g/f has only removable singularities.

Indeed a possible singularity only occurs at zeros of f ($=$ zeros of g). For $c \in Z(f)$ locally we can write

$$f(z) = (z-c)^{\underline{o_c(f)}} \tilde{f}(z) \qquad \tilde{f} \not\equiv 0$$

$$g(z) = (z-c)^{\underline{o_c(g)}} \tilde{g}(z)$$

equal!

hence locally near c; since $o_c(f) = o_c(g)$:

$$\frac{g(z)}{f(z)} = \underbrace{\frac{(z-c)^{o_c(f)}}{(z-c)^{o_c(g)}}}_{\equiv 1} \frac{\tilde{g}(z)}{\tilde{f}(z)} = \frac{\tilde{g}(z)}{\tilde{f}(z)} \qquad \neq 0!$$

but the rhs extends holo to $z = c$ by constr!

Moreover since $Z(g) = Z(f)$ by the previous formula g/f never vanishes

Since every entire fct which never vanishes is of the form e^h for some entire fct h (Cor 5.8)

$$\frac{g}{f} = e^h \quad \text{for some entire } h. \qquad \square$$